On the selection of the Bandwidth Parameter for the *k*-Chart

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Abstract

The k-chart, based on Support Vector Data Description, has received recent attention in the literature. We review four different methods for choosing the bandwidth parameter, s, when the k-chart is designed using the Gaussian kernel. We provide results of extensive Phase I and Phase II simulation studies varying the method of choosing the bandwidth parameter along with the size and distribution of sample data. In very limited cases the k-chart performed as desired. In general, we are unable to recommend the k-chart use in a Phase I or Phase II process monitoring study in its current form.

Keywords: One-class classification, Process Monitoring, Support Vector Data Description

1 Introduction

The k-chart [9] based on Support Vector Data Description ([10], [12]) has received recent attention in the literature. Support Vector Data Description (SVDD) is a One Class Classification (OCC) method that attempts to specify a boundary with the minimum volume around a distribution of data. There are several different types of OCC methods used in process monitoring and for a recent overview see Weese et al. [14]. When directly applied to the data, SVDD encapsulates data by fitting the minimum volume hypersphere around the data and when a kernel tranformation is used with SVDD, the boundary can take any flexible shape. Although Tax and Duin [10] did not present SVDD specifically as a process monitoring method, the supporting example in Tax et al. [13] is in essence a classic Phase I problem. In their motivating example engineers seek to establish baseline operating conditions for a machine, which is the goal of a Phase I anlaysis. It is important to note that the baseline dataset with which to establish the control boundary was known to be in control, which is not a common assumption in Phase I analysis. Sun and Tsung [9] noticed the similarity between SVDD and statistical process control and adapted SVDD as a control chart known as the k-chart. There have been several case studies published ([4], [3], [5]) using the k-chart. Subsequently, there is a wide range of guidance on the implementation of this chart in practice, primarily for Phase II applications where the goal is to monitor a process for changes. The goal of this work is to expand upon previous simulation studies and suggest how to choose the bandwidth of the Gaussian kernel function for k-chart performance. The paper is organized as follows. In Sections 2 and 3 we will give an overview of SVDD and the k-chart. In Section 4 we discuss the results of the simulations for the Phase I use of the k-chart and for the Phase II use in Section 5. Finally, we present recommendations to users regarding the use of the k-chart and discuss the best choice of s, the bandwidth parameter, for desired performance on any size data set.

2 Support Vector Data Description

Inspired by Support Vector Machines (SVM), SVDD is an unsupervised learning method used to give a description (or produce a boundary) around a data set. Whereas SVM separates classes by maximizing the margin (the distance between the closest objects of two classes), SVDD maximizes the minimum volume surrounding a data set and relies on user supplied parameters to determine how large the boundary should be. In SVM the boundary between the two classes is defined by only a few points of each class, called the support vectors. Similarly, in SVDD the boundary surrounding a data set is also defined only by the points farthest from the center of the data. These boundary defining points are referred to as the support vectors. To obtain the SVDD hypersphere, defined by a center μ and a radius R, we minimize R using

$$\begin{gathered} F(R, \mu, \xi_i) = R^2 + C \sum_i \xi_i \\ \text{s.t. } (x_i - \mu)^T (x_i - \mu) \leq R^2 + \xi_i \text{ and } \xi_i \geq 0, i = 1, ..., N, \end{gathered} \tag{1}$$

where each ξ_i is a slack variable that models the error for the i^{th} of N observations. C is a trade-off parameter which controls the size of ξ_i and the fraction of samples lying on or outside the boundary. Essentially C controls the trade-off between the complexity of the boundary and the error. The boundary defining support vectors are either going to be outside the boundary ($\xi_i > 0$) or on the boundary ($\xi_i = 0$). As C increases, the hypersphere will contain more data. When some outliers are expected (as in a typical process control situation) then C can be set as $C \leq \frac{1}{N \cdot \text{fraction outliers}}$. Since it is more intuitive to set the fraction of outliers than the value of C, Tax and Duin [11] defined the parameter ν as $\nu = \frac{1}{NC}$, or the fraction of outliers expected. From this point forward, we will discuss the choice of C by the specification of ν. The shape of the boundary found by SVDD can also be altered by employing a kernel transformation to the data prior to performing the optimization in equation (1). Although many different kernels can be used, Tax and Duin [10] shows that the Gaussian Kernel is preferred. Equation (2) shows the Gaussian kernel function, which requires the specification of the bandwidth parameter s.

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$$K(x,y) = exp\Big(-\frac{\|x-y\|^2}{s^2}\Big) \tag{2}$$

The simultaneous optimization of s and ν creates complexity when determining the appropriate SVDD boundary for a data set and there have been several recommendations on how to simultaneously choose these parameters to find the optimal boundary in the literature (e.g., [10], [11], [12], [1]). In general the larger the value of s, the larger the boundary will be and the choice of s is more important to the construction of the optimal boundary than ν. In fact, Brereton and Lloyd [1] and Tax and Duin [10] recommend choosing ν to be 0, optimize s and then adjust ν accordingly. Tax and Duin [11] introduce a method for generating "fake" outliers to surround a data set and using the Type I and Type II error rates of the samples and outliers accordingly to simultaneously optimize s and ν. In this paper we will specify the choice of ν, as is typically done in Phase I, and examine methods for determining s.

The choice of s for SVDD has been a topic of research and a particularly appealing method, due to its simplicity, is presented by Khazai et al. [7]. They recommend choosing s based on equation (3) where s_{ij}^2 is the sample variance of variable j for class i.

$$s = \Big(\sum_{i=1}^{N} s_{ij}^2\Big)^{1/2} \tag{3}$$

Tax and Duin [10] show that the upper bound of the error on a data set can be estimated using the number of support vectors (SV) divided by the sample size (N). In fact, one could iterate through values of s to find the value that gives the number of support vectors to generate a desired error rate for the baseline data set. Tax and Duin [12] also state that if the number of support vectors will not decrease sufficiently enough to obtain the error estimate desired, then a larger sample size is required.

Figure **??** shows the relationship between the bandwidth, s and the number of support vectors for a sample of N=100 observations from a p-variate t-distribution with d.f.=5 (p=2,

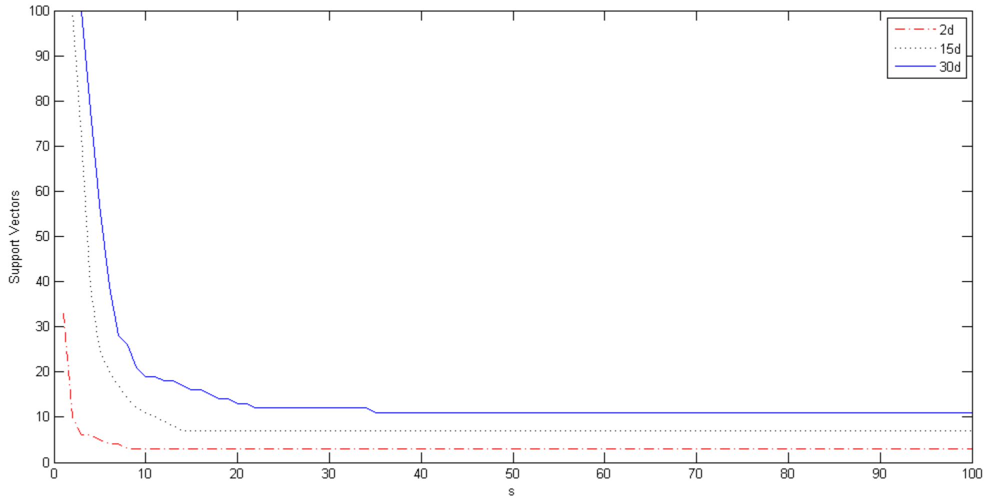
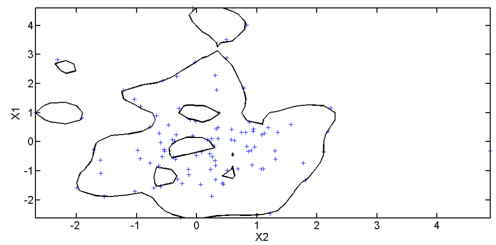


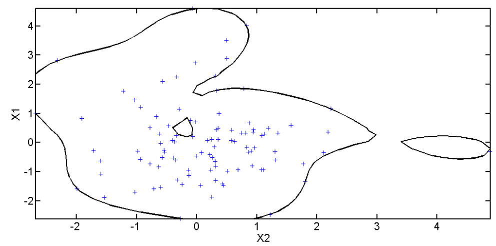
Figure 1: Plot of how the number of support vectors changes with the choice of the bandwidth parameter for data generated from the t-distribution (d.f=5) for N=100 and dimension (d)=2, 10, 50 and $\nu = 0$

15, 30). Because the number of support vectors affects the complexity of the boundary, it is very important that s is well chosen. As the value of s increases, the number of support vectors decrease and eventually level off for all dimensions. Additionally, higher dimensions require more support vectors. Figure 2 shows (for p=2) the boundaries produced for different values of s for $\nu = 0$. When s is small ($s = 1$)the boundary is overfit and $s = 5$ produces a boundary that is too large to describe the data. A value of s=3 seems to describe the distribution of this bivariate data well.

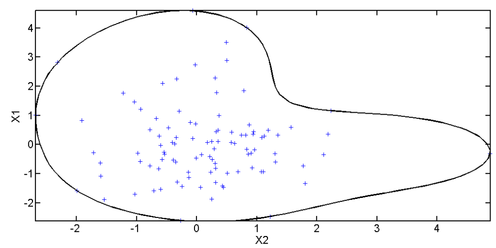
In addition to the bandwidth, s, the parameter ν also affects the shape of the boundary. Setting $s = 3$, Figure 3 displays the different boundaries when the parameter ν is varied. For each different value of ν=0, 0.01, 0.1, 0.25, the number of support vectors are 7, 7, 14, and 26 respectively. Additionally, notice that when $\nu = 0.01$ none of the n=100 points are outside the boundary. Recall, the parameter ν is an upper bound for the fraction of data outside of the boundary. In fact this boundary is not markedly different from the boundary when ν=0 (see Figures 3a and 3b). This is due to the fact that there is not a large enough sample size to attain the desired error. Since, $\#SV/N$ is the upper bound on the fraction



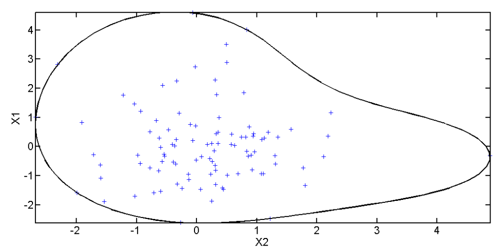
(a) s=1



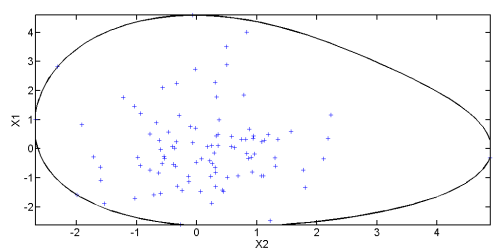
(b) s=2



(c) s=3



(d) s=4



(e) s=5

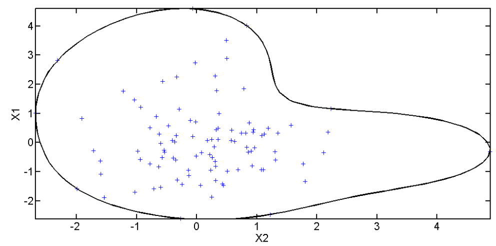
Figure 2: Different boundaries found for different values of s (ν=0) for N=100 and p=2 where x is generated from a t-distribution with d.f.=5

outside of the boundary. For N=100 and ν=0.01 there would only need to be one support vector. It is not possible to create a boundary with just one support vector. Geometrically, a minimum of two points are required to define the boundary if the boundary were a perfect sphere. As the shape become more complex, more points are required. As the number of support vectors required increases, the sample size must also increase to attain the desired fraction of outliers, thus setting the Type I error rate. Since each of the parameters ν and s change the boundary, either both must be optimized at one time as in Tax and Duin [11] or separately as in Brereton and Lloyd [1].

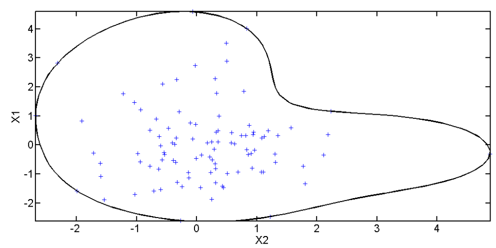
3 k Chart Overview

Sun and Tsung [9] were the first to recognize the natural extension of SVDD to a typical process monitoring problem. They used the kernel distance from the center of the data description as the monitoring statistic and the boundary generated by SVDD as the control chart limit. They noted that in cases where the data is not normally distributed, the k-chart might be preferable to the T^2 chart. In this early work, they do not provide guidance on the choice of the parameters, ν or s. Gani et al. [4] compared the performance of Sun and Tsung's [9] design of the k-chart with the k Nearest Neighbors-chart (kNN-chart) but did not give recommendations on the selection of ν or s.

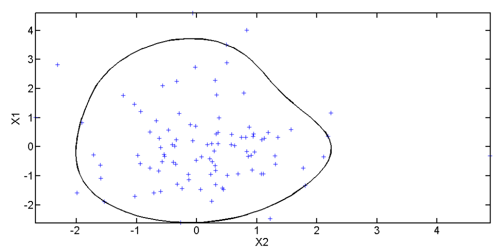
Ning and Tsung [8] updated the design of Sun and Tsung [9] using the outlier method Tax and Duin [11] to optimize ν and s by modifying weight given to Type I and Type II errors. The design of the k-chart recommended Ning and Tsung [8] did not use the boundary found by SVDD, but the $(1-\alpha)^{th}$ quantile of the bootstraped kernel distances using the optimum s value. Grasso et al. [5] used this design of the k-chart in comparison with fuzzy neural networks on a study of multimode process data. It should be noted that Gani et al. [4], Gani and Limam [3] and Grasso et al. [5] assumed an in-control reference sample and compared Phase II performance based on ARL. Ning and Tsung [8] suggested a leave-one-out validation procedure to find the Phase I boundary; however, only Phase II



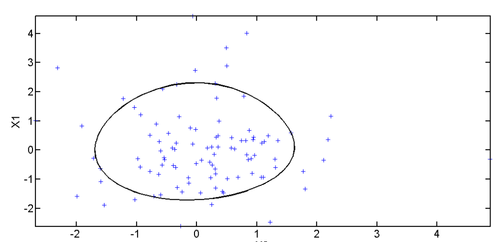
(a) ν=0



(b) ν=0.01



(c) ν=0.1



(d) ν=0.25

Figure 3: Different boundaries found for different values of ν ($s = 3$) for N=100 and p=2 where x is generated from a t-distribution with d.f.=5

simulation results, using the bootstrapped boundary are presented in their work. We do not include bootstrapped control limits in this study as it did not substantially improve performance. This is consistent with previous research, e.g. Jones and Woodall [6, p. 372] state that "The results of these simulation studies show that the bootstrap control charts do not perform substantially better than the standard method...."

While the procedure of Tax and Duin [11] and subsequently Ning and Tsung [8] enables a user to find values of s and ν, the computational demand needed to generate enough "fake" outliers to adequately fill the space is quite large. Tax and Duin [11] suggested a ratio of 1 observation to 250 outliers to find an adequate boundary for a dimension of p=10. Additionally, Ning and Tsung [8] showed that N=4000 is needed to attain desired in-control ARL for p=2. Using a rough estimate of Tax and Duin [11] ratio of 1 to 250 adjusted by 1/5 to account for the reduced dimension from p=10 to p=2 would necessitate 200,000 generated outliers.

The k-chart warrants further investigation for the following reasons: (1) the boundary is flexible (2) there is is no distributional assumption necessary (3) the PRTools Matlab toolbox [2] is available for free to implement the k-chart with minimal additional coding required (4) adaptation to kernels other than the Gaussian could allow for use with mixed data types. In the following sections we greatly expand upon previous simulation studies in the literature. We found the existing simulation studies to be limited, perhaps due to the computation requirements necessary to adequately assess the chart performance. Generating artificial outliers is computationally intensive and as such, Ning and Tsung [8] provide only a few simulation cases.

4 Phase I Simulation

Our goal was to investigate within and between sample Phase I performance using $m = 100$ different samples of size n. The simulation results that follow use 48 separate simulation cases of $m = 100$ Phase I reference samples each. To ease the computational burden, we did not directly use the method of Ning and Tsung [8]. We developed a method to emulate their choice of s. For a fixed value of ν, as s increases, the volume of the boundary surrounding the data increases, see Figure 2. Figure **??** shows that the number of support vectors rapidly decreases as the size of the bandwidth is increased and eventually levels off. Because the number of support vectors determines the boundary, it is clear that the boundary eventually stabilizes as s increases. The method developed by Ning and Tsung [8] finds a suitable boundary but requires the use of artificial outliers. Instead of implementing their method directly, we approximate their choice of s with a differencing approach. We observed that Ning and Tsung's [8] method located the (approximate) inflection point on the curve of the number of support vectors vs. s as shown in Figure 4. We developed a method to approximate their choice of s as follows:

1. Gather a sample $\mathbf{x} = \mathbf{x}_1, ..., \mathbf{x}_N$, of p-variate observations from a process.

2. Let SV_s be the number of support vectors found for $\mathbf{x}$ using SVDD for bandwidth $s = 1, ..., 500$.

3. Smooth the consecutive values of SVs using a moving average with a window of 2 calculating $S\bar{V}_s = \frac{SV_s + SV_{s+1}}{2}$ for $s = 1, ..., 499$

4. Compute the first differences of the $S\bar{V}_s$ values: $D_r = |S\bar{V}_{s+1} - S\bar{V}_s|$ for $s = 1, ..., 498$

5. The value of D_r that represents the first in the sequence such that $D_r < \delta$ (where δ is some nominal value) is the chosen value of s.

We refer to this method as the Ning and Tsung Approximation (NTA). Figure 4 displays the choice of s by the NTA method as well as the Khazai et al. [7] method (see Equation 3), labeled IEEE in Figure 4, for N=100 samples drawn from $N_p(\mathbf{0}, \mathbf{I})$ at various dimensions, p. One particular advantage of the NTA method is it can be used for data of any dimension. We compared our simulation results to those for the same case from Ning and Tsung [8] and for the case with an in-control reference sample of $N_p(\mathbf{0}, \mathbf{I})$ with $p = 2$ and $N = 500$ and

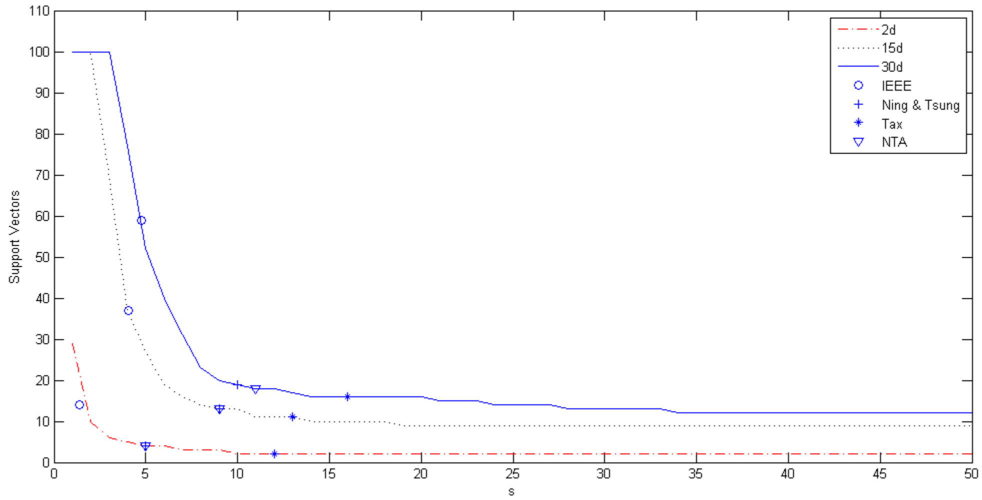


Figure 4: Comparison of the different bandwidth methods choice of s for $N = 100$ samples from $N(0, 1)$ data for different dimensions.

using an $\alpha = 0.01$ they reported an ARL_0=53.05 for what seems to be a single in-control reference sample. We found under the same conditions for 100 reference samples the average ARL_0=66.259 (see Table 8 in the Appendix). Using the approximating function was the only feasible way to perform the number of necessary simulations to quantify the effect of sample to sample variability on the k-chart performance when the method of Ning and Tsung [8] is used to determine s.

We compare the NTA method with three other methods for choosing s which we refer to as TAX, P and IEEE. The TAX method from Tax and Duin [10] simply uses the rule that the number of support vectors (SV) divided by N, $(\#SV/N)$, is an upper bound on the Type I error rate. As suggested by Tax and Duin [10], we set the value of s such that the quantity $\#SV/N$ is equal to the chosen value of the Type I error rate and then iterate through different values of s until the appropriate number of support vectors is found. In some cases the appropriate number of support vectors will be large because N is not large enough to accommodate the desired Type I error rate. The smallest number of possible support vectors to describe any data for which $p \geq 2$. Thus if $N = 100$ the smallest Type I error rate that can possibly be obtained is 0.02. To obtain a Type I error rate of 0.01, at least $N = 200$ is required, if the boundary is a perfect sphere. In our simulations we did not run the cases where the desired value of the Type I error rate could not be obtained. You will also see that sometimes even though N was apparently large enough relative to the desired Type I error rate, the boundary produced was clearly too large to be of use.

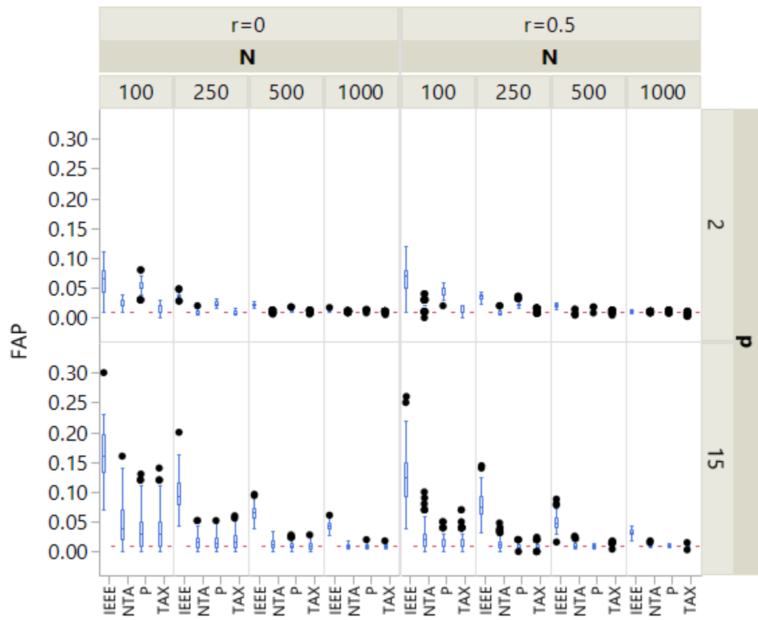
Khazai et al. [7] suggested taking the square root of the sum of the variance of each variable in the sample data as the estimate of s, as shown in equation (3). Lastly, we simply choose the value of s as the number of dimensions of the data, p. We refer to this method as P in the results.

4.1 Phase I Simulation Protocol

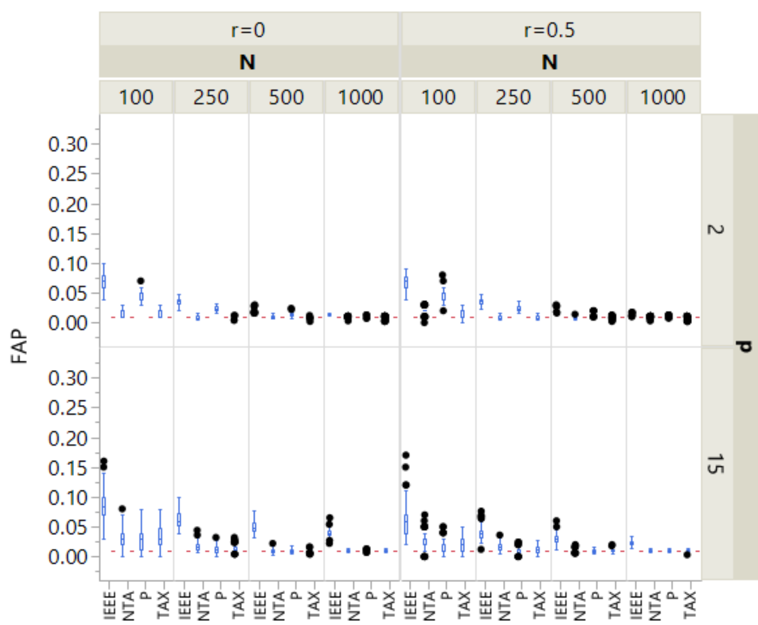
To compare the four methods for choosing s (NTA, TAX, IEEE and P) we varied the following parameters for the the in-control Phase I data:

- Distribution: $N_p(\mathbf{0}, \Sigma)$ or $LogNormal_p(\mathbf{0}, \Sigma)$ such that $\sigma_{ii} = \sigma_{jj}$ for all $i, j = 1, ..., p$.
- Dimension: p=2, 15
- Type I Error Rate: α=0.01, 0.1
- Sample Size: N=100, 250, 500, 1000
- Correlation: σ_{ij}=0, 0.5 for all $i, j = 1, ..., p, i \neq j$.

We compared 48 different simulation scenarios for each of the four methods for selecting s. For each of the 48 different combinations we generated $m = 100$ different Phase I samples to assess sample to sample performance. We quantified Phase I performance with False Alarm Probability (FAP) computed as $\frac{\text{the number of in-control points deemed out-of-control}}{N}$. Complete tabulated results can be seen Tables 1-6 in the Appendix. Figures 5 and 6 plot the FAP for each set if $m = 100$ in-control Phase I samples for α=0.01 and 0.1, respectively. For both figures we see improvement in the accuracy and precision of the FAP as N increases and better performance for $p = 2$ than $p = 15$ for a given sample size. We see that the

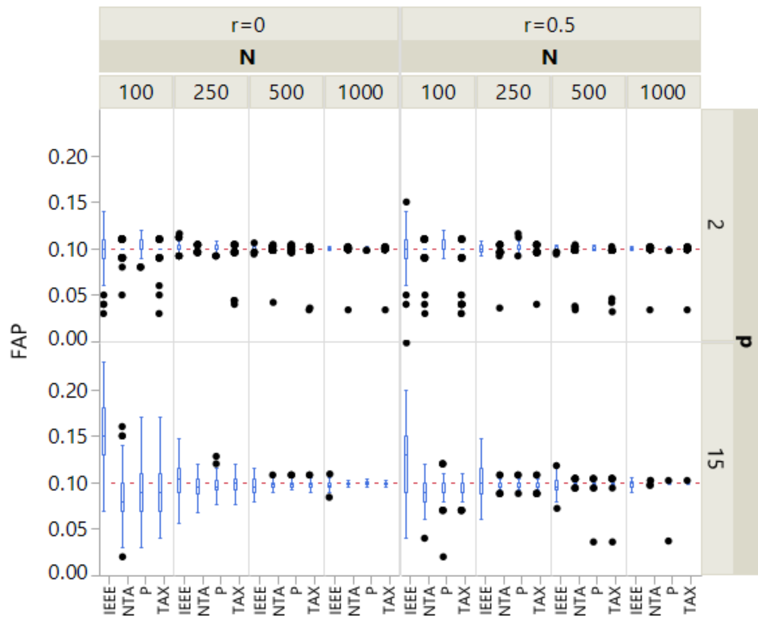


(a) p-variate Normal Distribution

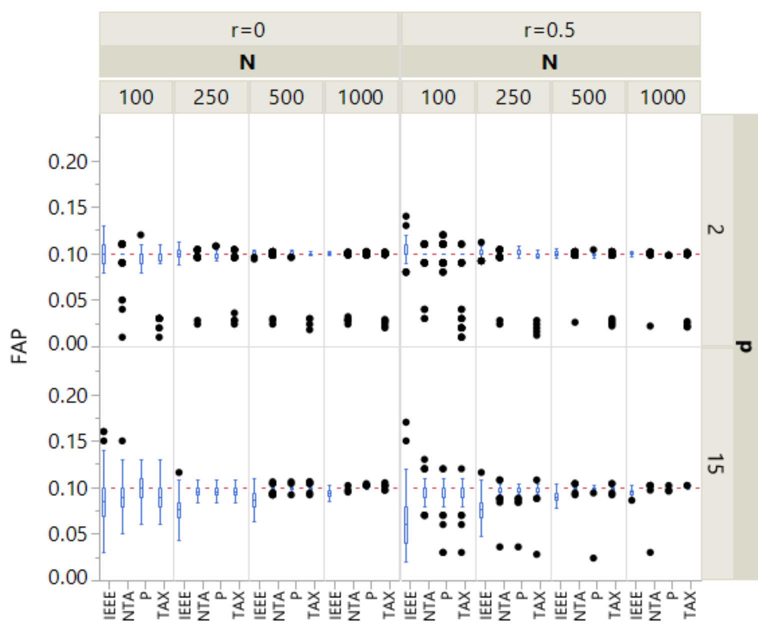


(b) p-variate LogNormal Distribution

Figure 5: FAP for $\alpha = 0.01$ for each bandwidth method. Note the dashed line is the targeted $\alpha = 0.01$.



(a) p-variate Normal Distribution



(b) p-variate LogNormal Distribution

Figure 6: FAP for $\alpha = 0.1$ for each bandwidth method. Note the dashed line is the targeted $\alpha = 0.1$.

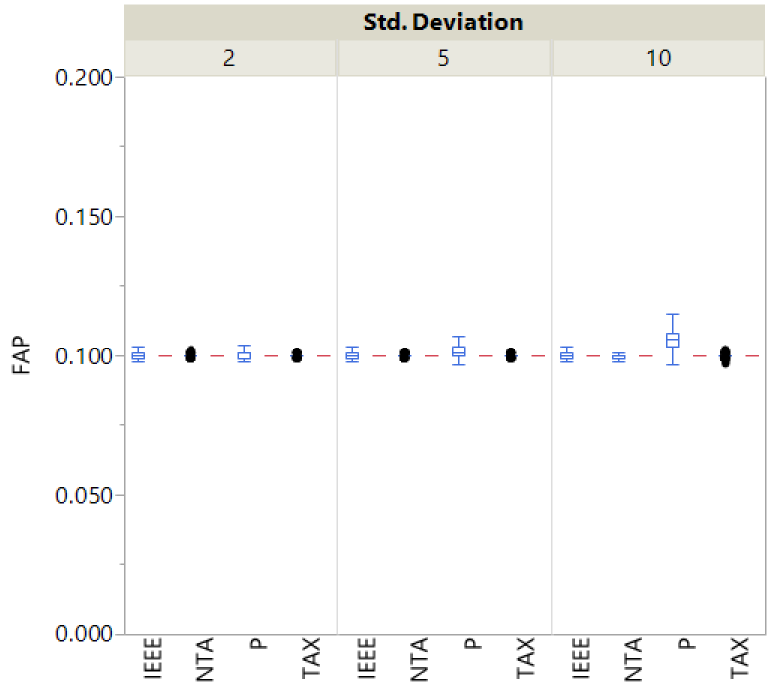


Figure 7: FAP for $N_p(\mathbf{0}, \Sigma)$ where N=1000, p=2, $\alpha = 0.1$ where σ_{ii}=2, 5 and 10 for each s method.

TAX and the NTA method perform similarly and that the IEEE method tends to overfit the data leading to a high FAP. Simply choosing p as the value of s works better when $p = 15$. Unsurprisingly, the larger Type I error probability results in better performance. The FAP does not seem to differ between the Normal and Lognormal distributions and improves slightly with correlation present. We speculate that the addition of correlation results in a lower volume boundary, which is better "filled" by the N observations.

The reference samples used to generate the simulation results shown in Figures 5 and 6 have unit variance. The performance of the P method for choosing s would be ill-advised if the generated data do not have unit variance. Figure 7 shows FAP results for N=1000, p=2 with r=0 and α=0.1 for $N_p(\mathbf{0}, \Sigma)$ for values of σ_{ii}=2, 5, and 10. Notice that as the variance increases the performance of the P method deteriorates. Thus it is important to standardize the data when selecting s according to $s = p$. The other methods have similar performance for σ=2 and 5. For σ=10, the IEEE method seems unchanged and the NTA and TAX methods show more variability.

4.2 Contaminated Reference Sample Results

Usually it is unknown if a Phase I reference sample is in-control or out-of-control. For this reason, we assessed the performance of the k-chart, when the reference sample is contaminated with some out-of-control observations.

For the in-control sample, we generated observations using a multivariate normal distribution with mean vector $\mu_{\mathbf{0}} = \mathbf{0}$ and covariance matrix $\mathbf{\Sigma_0} = \mathbf{I}$,. To generate out-of-control data, the mean vector is shifted $\mu_{\mathbf{1}} = \mu_{\mathbf{0}} + \delta$, where δ is the shift according to the non-centrality parameter $\gamma = \sqrt{\delta^T \mathbf{\Sigma_0^{-1}} \delta}$. We varied the shift size using $\gamma = 1, 2$ for a small shift, $\lambda = 4, 6$ for a medium shift, and $\lambda = 10, 20$ for a large shift. For any given shift γ, we randomly selected only one dimension of p to incur the shift. For example if the first dimension is randomly selected to be shifted, then $\mu_{\mathbf{1}} = [\mu_{\mathbf{1}} + \delta\ \mu_{\mathbf{2}}\ \mu_{\mathbf{3}}\ ...\ \mu_{\mathbf{p}}]$. The simulations were run using the following parameters.

- Distribution: $N_p(\mathbf{0}, \mathbf{\Sigma})$ where $\mathbf{\Sigma} = \mathbf{I}$
- Shift Size: $\lambda = 1, 2, 4, 6, 10, 20$
- Dimension: p=2, 15
- Sample Size: N=100, 500
- Correlation: $\sigma_{ij} = 0$
- Type I Error Rate: α=0.01, 0.1
- Percent Contamination: 5, 10, 25

In general, none of the bandwidth methods produced boundaries that were able to detect the out-of-control data. Figure 8 gives and example of the typical boundary found using the different methods. Here the out-of-control points are shown as circles. It is clear that none of the boundaries effectively identify the out-of-control observations. The results of the simulation further this conclusion. For example, when the percent of out-of-control points in the reference sample was set to be 25% the mean Type II error for all s methods, for any value of N, was about 25%. This implies that most of the outliers were deemed in-control. Complete simulation results for contaminated cases are available from the first author by request.

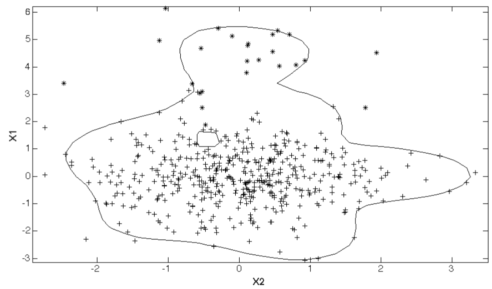
5 Phase II Simulation

Assuming there is an in-control reference sample, we conducted Phase II simulations using the same methods for finding s. For each of 100 in-control reference samples, we calculated the ARL_0 for 1000 Phase II samples and report the mean ARL_0 for each of the 100 Phase I samples. The simulations were conducted as follows:

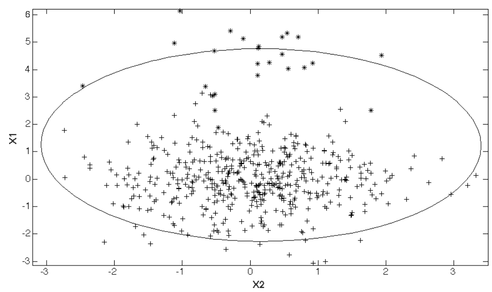
1. Take a sample of N in-control observations from the specified distribution.
2. For a set α, p and correlation, find the value of s using one of the four stated methods and establish a control limit.
3. Sample observations from the specified distribution and compare to the boundary. Repeat until an observation falls outside of the control limit, record the order of that observation.
4. Repeat step (3) 1000 times and calculate the ARL_0 conditioned on the reference sample.
5. Repeat steps (1)-(4) for each of 100 Phase I samples.

The following were used in the above simulations:

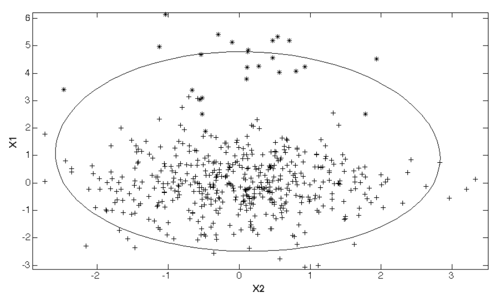
- Distribution: $N_p(\mathbf{0}, \Sigma)$ or $LogNormal_p(\mathbf{0}, \Sigma)$ such that $\sigma_{ii} = \sigma_{jj}$ for all $i, j = 1, ..., p$.



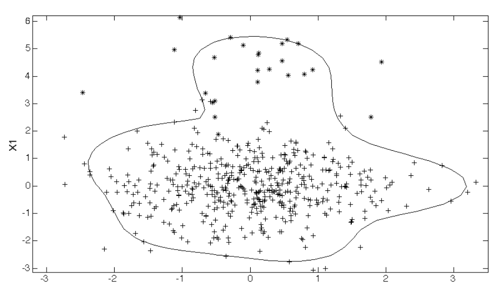
(a) IEEE, $s = 1.69$, Type I = 0.0337, Type II = 0.0360



(b) TAX, $s = 12$, Type I = 0.0274, Type II = 0.0260



(c) NTA, $s = 6$, Type I = 0.0316, Type II = 0.0300



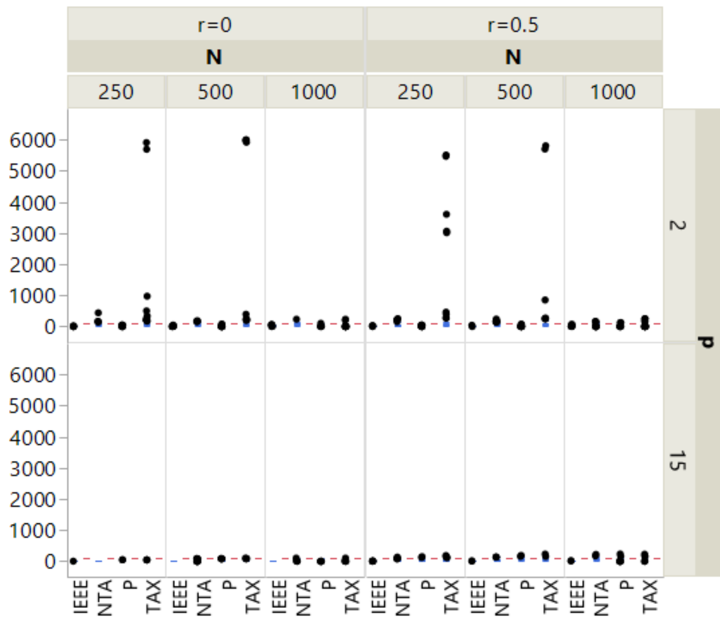
(d) P, $s = 2$, Type I = 0.0358, Type II = 0.0360

Figure 8: Boundaries for multivariate $N_p(\mathbf{0}, \mathbf{\Sigma})$, with $p = 2$, $\mathbf{\Sigma} = \mathbf{I}$, $\alpha = 0.05$, $r = 0$, $N = 500$ under data infected with 5% out-of-control points, $\gamma = 20$.

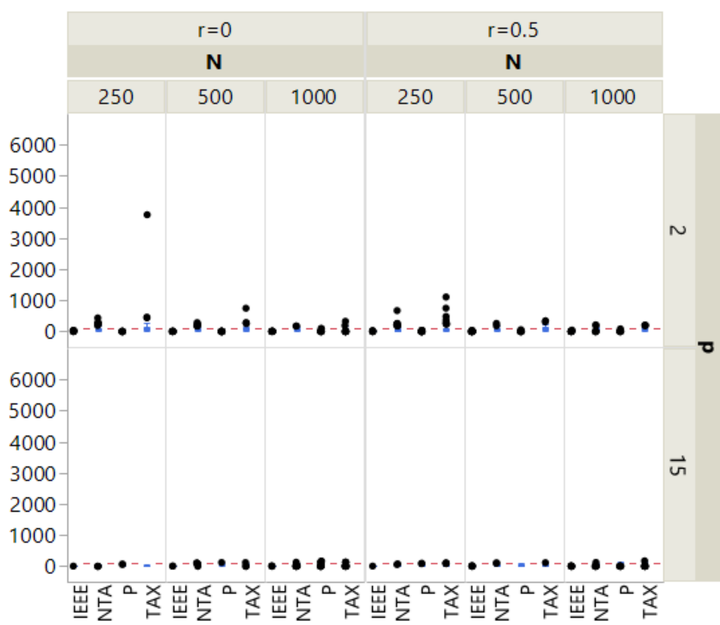
- Dimension: p=2, 15
- Type I Error Rate: α=0.01, 0.1
- Sample Size: N=100, 250, 500, 1000
- Correlation: σ_{ij}=0, 0.5 for all $i, j = 1, ..., p, i \neq j$.

Figures 9a and b show the conditioned ARL_0 values for $\alpha = 0.01$. For p=2 and smaller values of N the TAX method produces several large outliers. Figure 10, provides a better look at the majority of the conditional ARL_0 values. Disturbingly, the results indicate that the majority of the conditional ARL_0 values for most cases fall below the desired value of 100. Thus, the k-chart, regardless of the method for searching s would incur numerous false alarms in practice. None of the bandwidth methods achieve the targeted ARL_0 of 100, however performance is improved for larger values of N. It is interesting to note that performance seems to be more consistent, but still below target, when p=15. Table 8 in the Appendix gives the mean and standard deviation of the conditional ARL_0 values (AARL and SDARL). The TAX method has the best AARL but the largest SDARL in most cases. The NTA method produces slightly lower AARL values with smaller SDARL values than the TAX method, but the variation is still not acceptable. The P method works best when p=15.

Although ARL values are usually desired to be large in practice we consider the case with $\alpha = 0.1$ (ARL=10) for the sake of completeness. Figures 11a and b show that the Phase II performance is more consistent when α=0.1, but that the TAX method still produces large outliers when p=2 for the bivariate Normal Distribution. Figure 12 gives a better idea of the variability of the majority of the ARL_0 values when the data generated is from the Normal distribution. Table 7 in the Appendix show the the AARL and SDARL values. The TAX and NTA bandwidth methods produce AARL values closer to the target of 10, with the NTA method having smaller SDARL when p=2. The method of choosing s=p works best when $N > 100$, has the less variation than that of the TAX and NTA methods, but

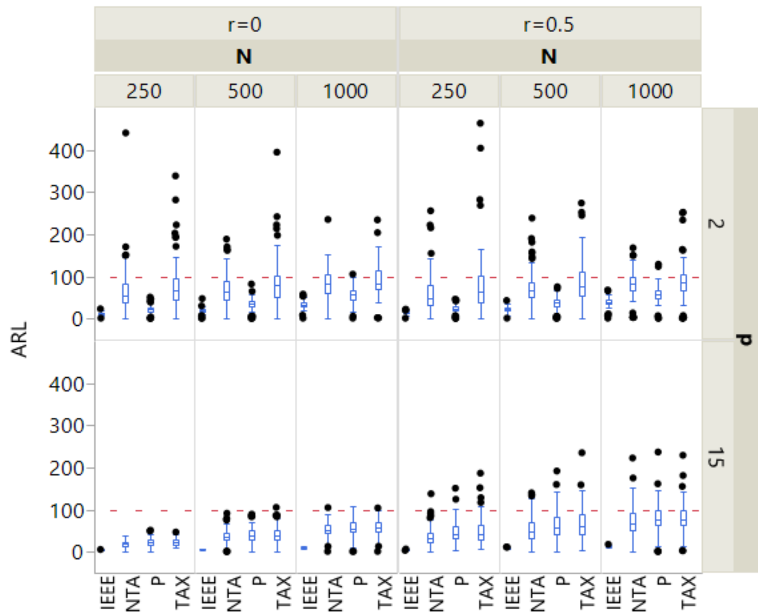


(a) p-variate Normal Distribution

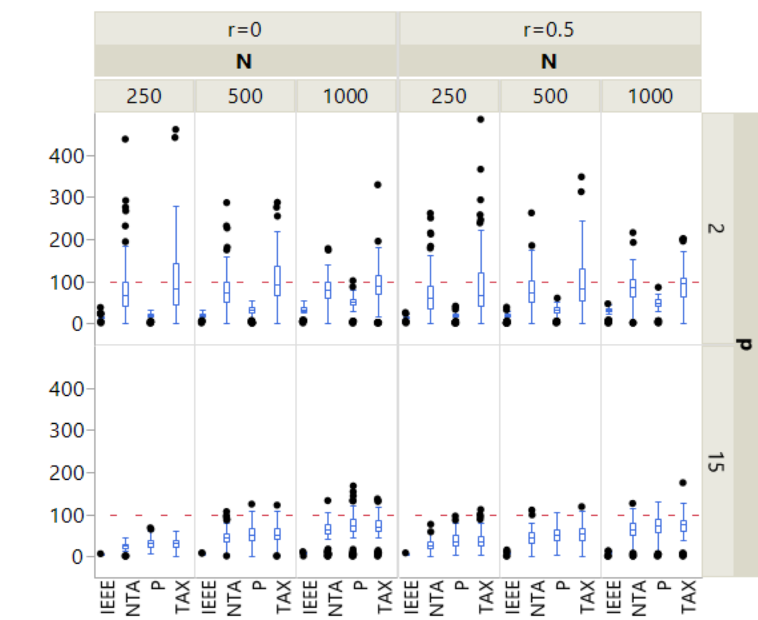


(b) p-variate LogNormal Distribution

Figure 9: ARL_0 for $\alpha = 0.01$ for each bandwidth method. Note the dashed line is the targeted $\mathrm{ARL}_0 = 100$

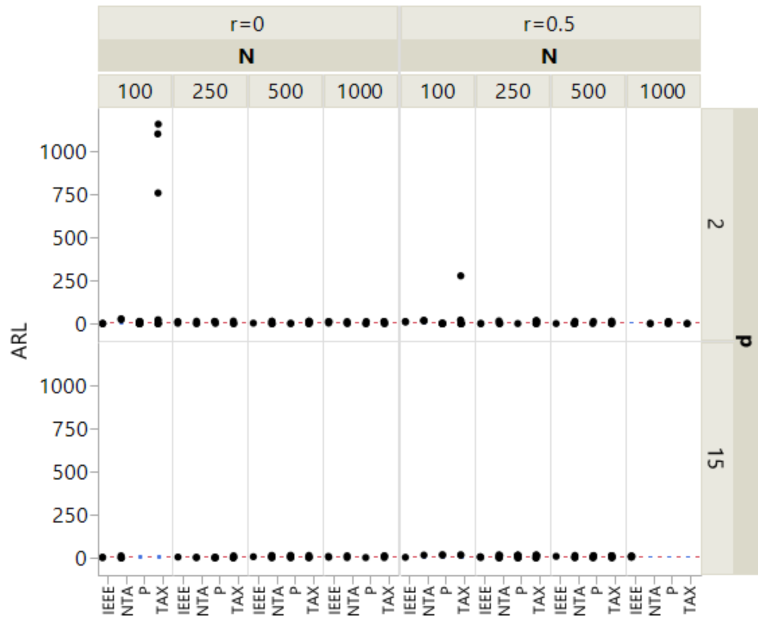


(a) p-variate Normal Distribution

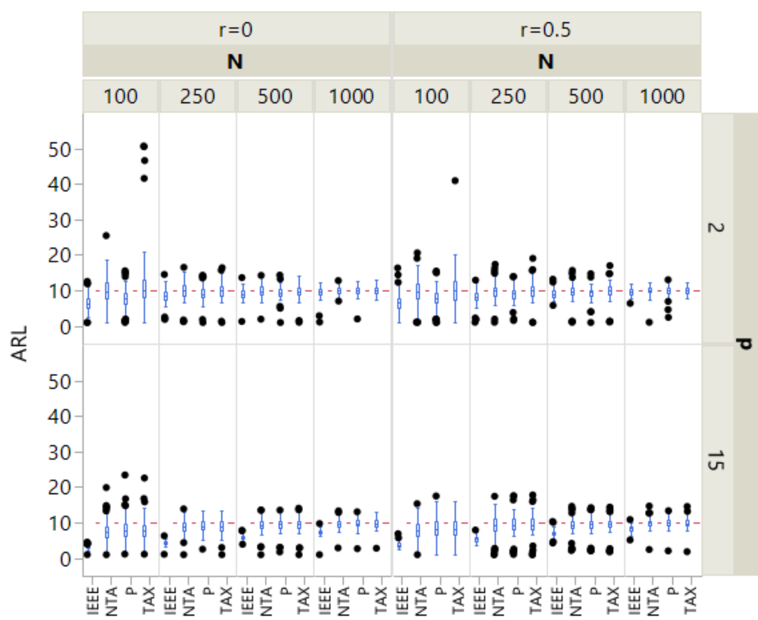


(b) p-variate LogNormal Distribution

Figure 10: Zoom of Figure 9. ARL_0 for $\alpha = 0.01$ for each bandwidth method. Note the dashed line is the targeted $\mathrm{ARL}_0 = 100$



(a) p-variate Normal Distribution



(b) p-variate LogNormal Distribution

Figure 11: ARL_0 for $\alpha = 0.1$ for each bandwidth method. Note the dashed line is the targeted $\mathrm{ARL}_0 = 10$

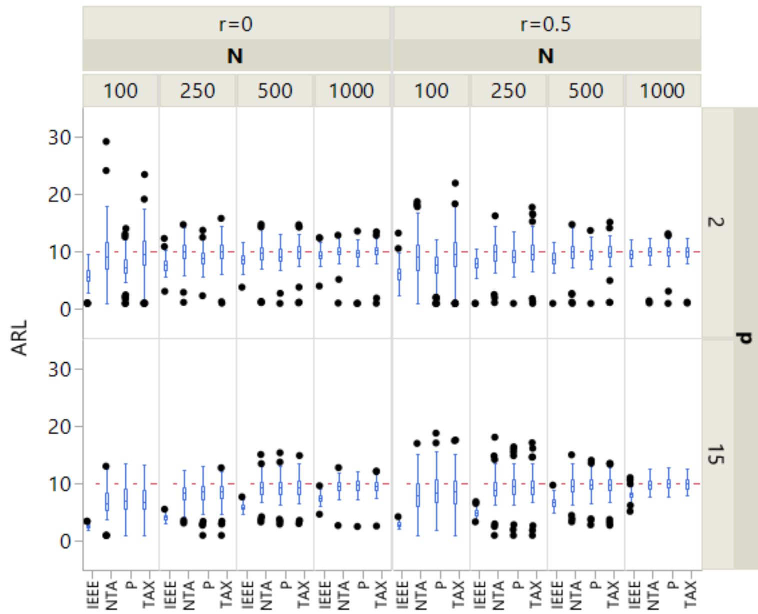


Figure 12: Zoom in of Figure 11a. ARL_0 for $\alpha = 0.1$ for each bandwidth method for samples from the p-variate Normal distribution. Note the dashed line is the targeted $\text{ARL}_0 = 10$

still produce charts with unacceptable in-control performance.

6 Recommendations

We have reviewed four different methods for choosing the bandwidth parameter, s, when the k-chart is designed using the Gaussian kernel function. When used in Phase I, all methods of determining s produce charts with FAP values near (or below) the desired Type I error rate when all observations are in-control. However, we found the chart to be incapable of distinguishing in-control and out-of-control observations when the reference sample is contaminated. Thus the use of the k-chart as a Phase I tool in its present form is not recommended.

In all cases studied, regardless of the method used for selecting s, the Phase II performance of the k-chart was unacceptable. In most cases, a majority of the k-charts produced ARL values below the desired level. This indicates that in practice, practitioners are likely to experience a large number of false alarms when using the k-chart in this form.

If using the k-chart, we would recommend using the NTA method or the method of Ning and Tsung [8] to choose the bandwidth parameter, s, in lower dimensions and the P method, of setting $s = P$, in higher dimensions. If one wishes to simply use the dimension, p as a choice of s, data should be transformed to unit variance prior finding the boundary. The IEEE method, while simple and robust to differing variances tends to overfit data and we would not recommend this method unless N is much greater than p as it only had close to acceptable performance when N=1000 and p=2. The TAX method has similar performance to the NTA method when correlation was present and N is large; however, there can be cases where the initial boundary is too large, to produce desired results thus caution must be taken when implementing the TAX method. We would also like to point out the large standard deviations of the the ARL_0 values for $\alpha = 0.01$ when the N to p ratio is small. In its current form, the k-chart performance is unacceptable for both Phase I and II monitoring.

The main problems with the k-chart are that the boundaries overfit for Phase II and cannot distinguish in- and out-of-control points in Phase I. Future research is necessary to find other chart designs that might improve performance of the k-chart. For example, work should be done to find a k-chart design that is less dependent on the choice of s, is more robust to outliers and requires a lower sample size. We do believe there is a lot of potential in one-class classification applications in Statistical Process Monitoring.

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7 Appendix

Table 1: Normal FAP

				IEEE				TAX				NTA				P			
r	α	p	N	Mean	S.D.	Min	Max	Mean	S.D.	Min	Max	Mean	S.D.	Min	Max	Mean	S.D.	Min	Max
0	0.01	2	100	0.063	0.023	0.010	0.110	0.017	0.006	0.000	0.030	0.023	0.007	0.010	0.040	0.054	0.010	0.030	0.080
		2	250	0.038	0.004	0.028	0.048	0.009	0.003	0.004	0.016	0.010	0.003	0.004	0.020	0.025	0.004	0.016	0.032
		2	500	0.022	0.002	0.016	0.028	0.010	0.001	0.006	0.012	0.010	0.001	0.006	0.012	0.014	0.002	0.010	0.018
		2	1000	0.012	0.001	0.010	0.017	0.010	0.001	0.005	0.011	0.010	0.001	0.008	0.012	0.011	0.001	0.008	0.014
		15	100	0.162	0.042	0.070	0.300	0.037	0.033	0.000	0.140	0.050	0.038	0.000	0.160	0.037	0.031	0.000	0.130
		15	250	0.097	0.026	0.044	0.200	0.019	0.014	0.000	0.060	0.017	0.013	0.000	0.052	0.016	0.013	0.000	0.052
		15	500	0.066	0.011	0.040	0.096	0.011	0.005	0.000	0.028	0.012	0.007	0.000	0.034	0.011	0.006	0.000	0.028
		15	1000	0.043	0.007	0.027	0.061	0.010	0.003	0.004	0.018	0.010	0.002	0.004	0.018	0.010	0.003	0.004	0.020
0	0.1	2	100	0.096	0.020	0.030	0.140	0.098	0.010	0.030	0.110	0.099	0.008	0.050	0.110	0.101	0.009	0.080	0.120
		2	250	0.102	0.005	0.092	0.116	0.098	0.011	0.040	0.104	0.100	0.002	0.096	0.104	0.101	0.004	0.092	0.108
		2	500	0.101	0.002	0.094	0.106	0.099	0.009	0.034	0.102	0.099	0.006	0.042	0.104	0.100	0.002	0.096	0.104
		2	1000	0.100	0.001	0.098	0.103	0.099	0.007	0.034	0.102	0.099	0.007	0.034	0.102	0.100	0.001	0.098	0.102
		15	100	0.153	0.037	0.070	0.260	0.092	0.027	0.040	0.170	0.086	0.028	0.020	0.160	0.092	0.027	0.030	0.170
		15	250	0.104	0.020	0.056	0.148	0.098	0.009	0.076	0.120	0.095	0.010	0.068	0.120	0.097	0.009	0.076	0.128
		15	500	0.097	0.009	0.080	0.116	0.098	0.004	0.090	0.108	0.098	0.004	0.090	0.108	0.099	0.004	0.092	0.108
		15	1000	0.097	0.004	0.084	0.109	0.099	0.001	0.096	0.102	0.099	0.002	0.096	0.103	0.099	0.002	0.095	0.104
0.5	0.01	2	100	0.065	0.020	0.010	0.120	0.012	0.007	0.000	0.020	0.020	0.007	0.000	0.040	0.045	0.008	0.020	0.060
		2	250	0.036	0.004	0.024	0.044	0.012	0.002	0.008	0.016	0.010	0.003	0.004	0.020	0.024	0.004	0.016	0.036
		2	500	0.020	0.002	0.014	0.026	0.010	0.001	0.004	0.012	0.010	0.002	0.004	0.014	0.013	0.002	0.008	0.018
		2	1000	0.012	0.001	0.008	0.015	0.009	0.002	0.002	0.011	0.010	0.001	0.008	0.011	0.010	0.001	0.008	0.013
		15	100	0.124	0.042	0.040	0.260	0.016	0.014	0.000	0.070	0.024	0.020	0.000	0.100	0.015	0.011	0.000	0.050
		15	250	0.078	0.021	0.032	0.144	0.010	0.005	0.000	0.024	0.013	0.009	0.000	0.048	0.010	0.004	0.000	0.020
		15	500	0.050	0.011	0.016	0.088	0.010	0.002	0.004	0.018	0.011	0.004	0.004	0.026	0.010	0.002	0.004	0.014
		15	1000	0.033	0.005	0.019	0.043	0.010	0.001	0.003	0.015	0.010	0.002	0.007	0.018	0.010	0.001	0.008	0.013
0.5	0.1	2	100	0.098	0.018	0.040	0.150	0.093	0.019	0.030	0.110	0.098	0.011	0.030	0.110	0.102	0.008	0.090	0.120
		2	250	0.100	0.004	0.092	0.108	0.099	0.006	0.040	0.104	0.099	0.007	0.036	0.104	0.101	0.004	0.092	0.116
		2	500	0.100	0.002	0.094	0.104	0.098	0.010	0.032	0.102	0.098	0.011	0.034	0.104	0.100	0.002	0.098	0.104
		2	1000	0.100	0.001	0.098	0.103	0.099	0.007	0.034	0.102	0.099	0.007	0.034	0.102	0.100	0.001	0.098	0.102
		15	100	0.125	0.042	0.040	0.260	0.093	0.010	0.070	0.110	0.091	0.012	0.040	0.120	0.093	0.013	0.020	0.120
		15	250	0.101	0.018	0.060	0.148	0.098	0.004	0.088	0.108	0.098	0.004	0.088	0.108	0.099	0.004	0.088	0.108
		15	500	0.097	0.008	0.072	0.118	0.099	0.007	0.036	0.104	0.099	0.002	0.094	0.104	0.098	0.007	0.036	0.104
		15	1000	0.098	0.004	0.089	0.106	0.100	0.001	0.098	0.102	0.100	0.001	0.097	0.102	0.099	0.006	0.037	0.102

Table 2: LogNormal FAP

				IEEE				TAX				NTA				P			
r	α	p	N	Mean	S.D.	Min	Max	Mean	S.D.	Min	Max	Mean	S.D.	Min	Max	Mean	S.D.	Min	Max
0	0.01	2	100	0.070	0.011	0.040	0.100	0.016	0.005	0.010	0.030	0.018	0.008	0.010	0.030	0.046	0.009	0.030	0.070
		2	250	0.036	0.005	0.020	0.048	0.008	0.002	0.004	0.012	0.009	0.003	0.004	0.016	0.024	0.004	0.016	0.032
		2	500	0.023	0.003	0.016	0.030	0.010	0.001	0.002	0.012	0.010	0.002	0.008	0.016	0.015	0.003	0.008	0.024
		2	1000	0.014	0.001	0.011	0.017	0.009	0.002	0.002	0.011	0.010	0.001	0.003	0.012	0.011	0.001	0.007	0.013
		15	100	0.089	0.027	0.030	0.160	0.032	0.020	0.000	0.080	0.034	0.017	0.000	0.080	0.029	0.019	0.000	0.080
		15	250	0.064	0.014	0.040	0.100	0.014	0.006	0.004	0.032	0.018	0.006	0.008	0.044	0.013	0.006	0.000	0.032
		15	500	0.051	0.010	0.032	0.078	0.011	0.003	0.004	0.016	0.011	0.003	0.002	0.022	0.010	0.003	0.004	0.018
		15	1000	0.040	0.006	0.022	0.065	0.010	0.001	0.007	0.014	0.010	0.001	0.007	0.014	0.010	0.001	0.007	0.013
0	0.1	2	100	0.100	0.009	0.080	0.130	0.091	0.024	0.010	0.110	0.098	0.014	0.010	0.110	0.100	0.008	0.080	0.120
		2	250	0.100	0.004	0.088	0.112	0.097	0.014	0.024	0.104	0.098	0.011	0.024	0.104	0.099	0.003	0.092	0.108
		2	500	0.100	0.002	0.094	0.104	0.097	0.015	0.018	0.102	0.098	0.012	0.024	0.102	0.100	0.002	0.096	0.104
		2	1000	0.100	0.001	0.098	0.103	0.095	0.018	0.020	0.101	0.096	0.016	0.024	0.101	0.100	0.001	0.098	0.102
		15	100	0.089	0.027	0.030	0.160	0.092	0.015	0.060	0.130	0.093	0.017	0.050	0.150	0.096	0.014	0.060	0.130
		15	250	0.078	0.014	0.044	0.116	0.097	0.005	0.084	0.108	0.096	0.005	0.084	0.108	0.097	0.005	0.084	0.108
		15	500	0.087	0.009	0.064	0.110	0.099	0.002	0.092	0.106	0.099	0.003	0.092	0.106	0.099	0.002	0.092	0.106
		15	1000	0.094	0.004	0.085	0.103	0.100	0.001	0.097	0.105	0.099	0.001	0.095	0.102	0.100	0.001	0.098	0.104
0.5	0.01	2	100	0.067	0.011	0.040	0.090	0.015	0.006	0.000	0.030	0.020	0.007	0.000	0.030	0.047	0.011	0.020	0.080
		2	250	0.037	0.005	0.024	0.048	0.010	0.003	0.004	0.016	0.009	0.003	0.004	0.016	0.025	0.004	0.016	0.036
		2	500	0.023	0.002	0.016	0.030	0.010	0.002	0.002	0.012	0.010	0.001	0.006	0.014	0.015	0.002	0.010	0.020
		2	1000	0.014	0.001	0.010	0.017	0.009	0.003	0.002	0.011	0.010	0.001	0.003	0.011	0.011	0.001	0.008	0.014
		15	100	0.060	0.027	0.020	0.170	0.018	0.012	0.000	0.050	0.027	0.015	0.000	0.070	0.017	0.012	0.000	0.050
		15	250	0.041	0.011	0.012	0.076	0.011	0.006	0.000	0.028	0.015	0.006	0.004	0.036	0.010	0.005	0.000	0.024
		15	500	0.031	0.007	0.012	0.060	0.011	0.003	0.004	0.020	0.011	0.003	0.006	0.020	0.010	0.002	0.004	0.016
		15	1000	0.023	0.005	0.013	0.035	0.010	0.001	0.003	0.014	0.010	0.001	0.007	0.014	0.010	0.001	0.008	0.014
0.5	0.1	2	100	0.103	0.011	0.080	0.140	0.087	0.029	0.010	0.110	0.098	0.014	0.030	0.110	0.100	0.008	0.080	0.120
		2	250	0.101	0.004	0.092	0.112	0.095	0.018	0.012	0.104	0.098	0.011	0.024	0.104	0.100	0.003	0.096	0.108
		2	500	0.100	0.002	0.096	0.106	0.096	0.016	0.022	0.102	0.099	0.007	0.026	0.102	0.100	0.002	0.096	0.104
		2	1000	0.100	0.001	0.097	0.102	0.096	0.017	0.021	0.101	0.099	0.008	0.022	0.101	0.100	0.001	0.098	0.101
		15	100	0.064	0.025	0.020	0.170	0.094	0.012	0.030	0.120	0.095	0.011	0.070	0.130	0.093	0.014	0.030	0.120
		15	250	0.076	0.013	0.048	0.116	0.097	0.008	0.028	0.108	0.097	0.008	0.036	0.108	0.097	0.007	0.036	0.104
		15	500	0.090	0.006	0.078	0.104	0.099	0.002	0.092	0.104	0.099	0.002	0.092	0.104	0.098	0.008	0.024	0.102
		15	1000	0.095	0.003	0.086	0.103	0.100	0.001	0.098	0.102	0.099	0.007	0.030	0.102	0.100	0.001	0.096	0.102

Table 3: Normal SV

				IEEE				TAX				NTA				P			
r	α	p	N	Mean	S.D.	Min	Max	Mean	S.D.	Min	Max	Mean	S.D.	Min	Max	Mean	S.D.	Min	Max
0	0.01	2	100	16.41	1.63	12	21	3.00	0.53	2	5	4.23	0.79	3	6	9.91	1.45	7	14
		2	250	21.63	2.06	16	27	3.85	0.69	3	6	4.45	0.83	3	7	12.07	1.58	9	19
		2	500	25.74	2.45	20	35	6.57	0.57	5	8	7.30	0.88	6	10	13.59	1.48	11	17
		2	1000	30.13	2.56	25	35	11.47	0.54	10	13	12.15	0.86	11	14	18.06	1.81	14	22
		15	100	39.77	2.64	34	47	10.60	2.02	6	16	12.08	2.33	7	18	10.07	1.88	6	15
		15	250	60.29	3.85	51	70	11.77	2.16	7	17	12.86	2.43	7	20	11.28	1.97	7	16
		15	500	78.53	3.84	68	94	13.06	2.39	8	20	14.44	2.65	8	21	12.76	2.11	9	20
		15	1000	101.40	4.77	90	118	17.33	2.08	13	23	18.55	2.44	14	25	16.96	1.86	13	23
0	0.1	2	100	18.11	1.60	15	22	11.25	0.54	10	13	11.62	0.66	10	13	14.28	1.30	12	18
		2	250	33.86	1.63	31	37	26.17	0.60	25	27	26.44	0.62	25	28	29.35	1.19	27	33
		2	500	59.29	1.69	55	63	51.08	0.53	50	52	51.48	0.69	50	53	54.15	1.09	52	56
		2	1000	109.31	1.64	106	114	101.24	0.62	100	103	101.64	0.84	100	104	104.25	1.12	102	107
		15	100	39.77	2.64	34	47	14.74	1.47	12	18	15.66	1.39	13	19	14.56	1.39	12	18
		15	250	60.70	3.77	51	70	29.65	1.57	27	34	30.36	1.73	27	35	29.60	1.48	27	33
		15	500	84.88	4.15	73	97	54.14	1.41	51	58	54.54	1.49	51	59	54.10	1.29	51	58
		15	1000	134.44	4.50	124	147	104.34	1.50	101	110	104.84	1.50	101	110	104.25	1.59	101	111
0.5	0.01	2	100	14.77	1.56	10	18	2.47	0.54	2	4	4.01	1.01	2	6	8.62	1.20	6	12
		2	250	18.83	1.93	14	24	4.17	0.45	3	5	4.89	0.90	3	8	11.23	1.29	8	14
		2	500	22.22	2.08	18	28	6.30	0.50	5	7	7.26	0.93	5	10	12.68	1.37	10	16
		2	1000	26.01	2.49	21	32	11.06	0.55	10	12	12.08	0.77	11	14	17.37	1.45	15	21
		15	100	34.21	2.95	27	42	4.65	1.59	2	9	6.71	2.67	3	16	4.52	1.31	2	9
		15	250	49.71	3.79	42	62	5.85	1.60	4	11	7.97	3.19	4	16	5.68	1.12	4	8
		15	500	61.53	3.83	52	71	7.96	1.57	6	14	9.94	3.32	6	19	7.78	1.09	6	10
		15	1000	76.94	4.23	67	87	13.22	1.84	10	19	14.84	3.30	11	26	12.74	1.11	11	16
0.5	0.1	2	100	17.07	1.42	14	21	10.95	0.48	10	12	11.42	0.65	10	13	13.56	1.20	12	17
		2	250	32.33	1.54	28	36	26.11	0.51	25	27	26.35	0.66	25	28	28.52	1.08	26	31
		2	500	57.22	1.45	54	61	51.03	0.54	50	53	51.38	0.63	50	53	53.42	1.05	52	57
		2	1000	109.31	1.64	106	114	101.24	0.62	100	103	101.64	0.84	100	104	104.25	1.12	102	107
		15	100	34.21	2.95	27	42	12.01	1.01	10	15	12.78	1.46	10	18	12.04	1.05	10	15
		15	250	51.39	3.60	44	62	27.11	0.91	26	30	27.75	1.24	26	32	27.07	0.90	26	30
		15	500	74.43	3.46	65	84	52.00	1.04	50	57	52.67	1.47	51	57	52.05	0.86	50	54
		15	1000	123.92	3.92	114	132	102.01	0.97	100	106	102.73	1.39	101	108	101.89	0.92	100	105

Table 4: LogNormal SV

				IEEE				TAX				NTA				P			
r	α	p	N	Mean	S.D.	Min	Max	Mean	S.D.	Min	Max	Mean	S.D.	Min	Max	Mean	S.D.	Min	Max
0	0.01	2	100	13.70	1.88	9	20	3.03	0.22	2	4	3.63	1.06	3	9	9.44	1.31	7	13
		2	250	18.97	2.11	13	24	3.25	0.48	3	5	3.80	1.10	3	7	12.49	1.76	8	16
		2	500	23.69	1.81	20	28	6.10	0.33	5	7	7.17	1.39	6	11	15.27	2.10	12	21
		2	1000	29.33	2.31	25	36	11.23	0.65	10	12	12.24	1.16	10	16	19.34	2.05	15	26
		15	100	34.08	2.37	28	39	7.74	1.44	5	11	9.34	1.85	6	14	7.47	1.32	5	11
		15	250	56.42	3.51	47	63	8.80	1.79	5	14	10.24	2.13	6	18	8.35	1.36	5	12
		15	500	82.81	4.23	72	94	10.08	1.64	7	14	11.27	2.15	7	17	9.92	1.39	7	14
		15	1000	122.59	4.70	105	133	14.32	1.52	12	19	15.20	1.90	12	21	14.03	1.33	12	18
0	0.1	2	100	15.61	1.45	13	20	11.04	0.62	10	12	11.39	0.68	10	13	13.33	1.01	12	16
		2	250	30.75	1.49	27	35	26.08	0.66	25	27	26.16	0.60	25	28	28.32	0.91	27	30
		2	500	56.00	1.44	54	59	51.04	0.68	50	53	51.21	0.57	50	52	53.30	1.13	51	57
		2	1000	106.03	1.25	103	110	101.06	0.63	100	102	101.18	0.63	100	103	103.40	0.99	101	106
		15	100	34.08	2.37	28	39	12.90	1.14	11	16	13.57	1.36	10	18	12.80	1.24	10	17
		15	250	56.39	3.49	47	63	27.59	1.13	26	31	27.85	1.28	26	32	27.56	1.11	26	31
		15	500	84.19	4.21	75	94	52.83	1.20	51	56	53.04	1.20	51	56	52.76	1.18	51	56
		15	1000	135.38	4.04	124	145	102.82	1.21	101	107	102.95	1.14	101	106	102.78	1.25	101	107
0.5	0.01	2	100	13.32	1.68	8	16	2.77	0.49	2	4	3.71	0.82	2	6	8.95	1.68	5	13
		2	250	18.58	2.06	13	24	3.62	0.69	3	5	3.97	0.90	3	7	12.63	1.74	9	17
		2	500	24.57	1.82	20	29	6.26	0.58	5	7	7.03	1.05	6	10	15.47	2.06	11	21
		2	1000	31.63	2.65	26	38	11.19	0.72	10	12	11.97	0.94	10	15	19.99	2.46	15	28
		15	100	27.35	3.16	16	35	5.81	1.53	3	11	7.22	2.03	4	13	5.32	1.15	3	8
		15	250	45.56	3.26	37	53	6.95	1.82	3	11	8.86	2.31	4	15	6.71	1.37	4	10
		15	500	67.24	4.86	45	79	9.36	1.74	6	15	10.67	2.17	7	17	8.99	1.11	7	12
		15	1000	100.51	5.45	86	111	13.98	1.61	10	18	15.11	1.93	12	22	13.86	1.17	12	18
0.5	0.1	2	100	15.50	1.62	11	20	10.96	0.58	10	12	11.29	0.67	10	13	13.03	1.06	11	17
		2	250	30.40	1.27	28	34	25.96	0.58	25	27	26.18	0.58	25	27	28.11	0.95	26	30
		2	500	55.58	1.53	52	59	51.02	0.59	50	52	51.07	0.64	50	53	53.04	1.01	51	56
		2	1000	105.85	1.35	102	110	100.92	0.54	100	102	101.18	0.59	100	102	103.08	1.02	101	106
		15	100	27.36	3.13	17	35	12.10	0.92	10	16	12.64	1.05	11	16	12.04	1.00	10	15
		15	250	45.86	3.20	39	53	26.97	0.95	25	30	27.17	0.92	25	29	26.93	0.98	25	30
		15	500	71.55	3.92	60	80	52.05	1.03	50	56	52.23	1.11	50	57	51.98	0.83	50	54
		15	1000	120.50	3.79	111	132	101.92	0.92	100	105	102.17	1.04	100	105	101.84	0.90	100	105

Table 5: Normal s

				IEEE				TAX				NTA			
r	$alpha$	p	N	Mean	S.D.	Min	Max	Mean	S.D.	Min	Max	Mean	S.D.	Min	Max
0	0.01	2	100	1.423	0.072	1.262	1.642	12.25	0.61	12	15	5.040	1.333	4	12
		2	250	1.418	0.045	1.277	1.522	12.20	0.57	12	15	5.690	1.662	4	12
		2	500	1.416	0.031	1.326	1.496	12.35	0.85	9	16	5.250	1.872	3	14
		2	1000	1.417	0.022	1.361	1.481	12.33	1.04	6	16	5.310	1.952	3	13
		15	100	3.881	0.074	3.687	4.055	13.75	1.62	12	19	10.270	1.797	7	15
		15	250	3.876	0.047	3.732	3.975	14.13	1.85	12	21	11.150	1.794	7	17
		15	500	3.876	0.034	3.796	3.953	14.75	2.11	12	21	11.800	1.781	8	17
		15	1000	3.874	0.023	3.818	3.934	14.35	1.99	12	20	11.420	1.810	8	17
0	0.1	2	100	1.423	0.072	1.262	1.642	12.25	0.61	12	15	5.040	1.333	4	12
		2	250	1.418	0.045	1.277	1.522	12.20	0.57	12	15	5.690	1.662	4	12
		2	500	1.416	0.031	1.326	1.496	12.35	0.85	9	16	5.250	1.872	3	14
		2	1000	1.417	0.022	1.361	1.481	12.33	1.04	6	16	5.310	1.952	3	13
		15	100	3.881	0.074	3.687	4.055	13.75	1.62	12	19	10.270	1.797	7	15
		15	250	3.876	0.047	3.732	3.975	14.13	1.85	12	21	11.150	1.794	7	17
		15	500	3.876	0.034	3.796	3.953	14.75	2.11	12	21	11.800	1.781	8	17
		15	1000	3.874	0.023	3.818	3.934	14.35	1.99	12	20	11.420	1.810	8	17
0.5	0.01	2	100	1.422	0.081	1.226	1.644	12.10	0.44	12	14	4.840	1.117	3	9
		2	250	1.418	0.051	1.283	1.553	12.08	0.34	12	14	5.170	1.240	4	11
		2	500	1.417	0.036	1.316	1.497	12.27	1.17	6	16	5.060	1.332	4	12
		2	1000	1.417	0.026	1.343	1.469	12.05	1.32	7	16	5.390	1.693	3	11
		15	100	3.893	0.154	3.527	4.285	15.42	1.89	12	21	11.960	2.365	7	18
		15	250	3.871	0.084	3.709	4.110	15.70	1.77	12	19	12.440	2.071	8	18
		15	500	3.876	0.073	3.718	4.078	15.72	1.72	12	20	12.660	2.185	8	18
		15	1000	3.877	0.051	3.739	3.988	15.40	1.98	12	20	12.620	2.201	8	18
0.5	0.1	2	100	1.422	0.081	1.226	1.644	12.10	0.44	12	14	4.840	1.117	3	9
		2	250	1.418	0.051	1.283	1.553	12.08	0.34	12	14	5.170	1.240	4	11
		2	500	1.417	0.036	1.316	1.497	12.27	1.17	6	16	5.060	1.332	4	12
		2	1000	1.417	0.022	1.361	1.481	12.33	1.04	6	16	5.310	1.952	3	13
		15	100	3.893	0.154	3.527	4.285	15.42	1.89	12	21	11.960	2.365	7	18
		15	250	3.871	0.084	3.709	4.110	15.70	1.77	12	19	12.440	2.071	8	18
		15	500	3.876	0.073	3.718	4.078	15.72	1.72	12	20	12.660	2.185	8	18
		15	1000	3.877	0.051	3.739	3.988	15.40	1.98	12	20	12.620	2.201	8	18

Table 6: LogNormal s

				IEEE				TAX				NTA			
r	α	p	N	Mean	S.D.	Min	Max	Mean	S.D.	Min	Max	Mean	S.D.	Min	Max
0	0.01	2	100	1.429	0.182	1.099	2.106	12.080	0.394	12	15	5.090	1.223	3	10
		2	250	1.419	0.128	1.202	1.877	12.180	0.609	12	16	5.970	1.521	4	13
		2	500	1.408	0.092	1.182	1.633	12.080	0.506	9	14	5.670	1.371	3	13
		2	1000	1.418	0.075	1.252	1.616	11.890	1.497	7	18	5.570	1.610	3	12
		15	100	3.891	0.236	3.377	4.515	13.960	1.891	12	21	10.130	1.555	7	14
		15	250	3.877	0.148	3.529	4.362	14.490	1.789	12	20	11.250	1.772	8	18
		15	500	3.882	0.115	3.645	4.365	14.910	2.050	12	21	12.080	1.745	9	18
		15	1000	3.879	0.070	3.727	4.128	14.300	1.834	12	20	11.850	1.572	9	17
0	0.1	2	100	1.429	0.182	1.099	2.106	12.080	0.394	12	15	5.090	1.223	3	10
		2	250	1.419	0.128	1.202	1.877	12.180	0.609	12	16	5.970	1.521	4	13
		2	500	1.418	0.100	1.224	1.742	12.140	0.551	11	15	5.720	1.939	3	14
		2	1000	1.418	0.075	1.252	1.616	11.890	1.497	7	18	5.570	1.610	3	12
		15	100	3.891	0.236	3.377	4.515	13.960	1.891	12	21	10.130	1.555	7	14
		15	250	3.877	0.148	3.529	4.362	14.490	1.789	12	20	11.250	1.772	8	18
		15	500	3.882	0.115	3.645	4.365	14.910	2.050	12	21	12.080	1.745	9	18
		15	1000	3.879	0.070	3.727	4.128	14.300	1.834	12	20	11.850	1.572	9	17
0.5	0.01	2	100	1.423	0.220	1.038	2.176	12.110	0.424	12	15	5.080	1.361	3	12
		2	250	1.423	0.144	1.181	1.828	12.410	0.830	12	16	5.830	1.415	4	11
		2	500	1.416	0.106	1.185	1.810	12.260	1.143	8	16	6.020	2.030	4	14
		2	1000	1.409	0.080	1.213	1.650	11.980	1.463	6	16	5.920	1.704	4	12
		15	100	3.901	0.418	3.134	5.221	13.780	1.667	12	18	10.480	1.720	7	15
		15	250	3.837	0.230	3.288	4.640	15.330	1.985	12	20	11.630	1.963	9	17
		15	500	3.856	0.224	3.440	5.313	15.270	2.201	12	21	12.490	1.789	9	17
		15	1000	3.865	0.145	3.549	4.239	15.550	2.467	12	23	12.580	1.597	9	17
0.5	0.1	2	100	1.423	0.220	1.038	2.176	12.110	0.424	12	15	5.080	1.361	3	12
		2	250	1.423	0.144	1.181	1.828	12.410	0.830	12	16	5.830	1.415	4	11
		2	500	1.416	0.106	1.185	1.810	12.260	1.143	8	16	6.020	2.030	4	14
		2	1000	1.409	0.080	1.213	1.650	11.980	1.463	6	16	5.920	1.704	4	12
		15	100	3.901	0.418	3.134	5.221	13.780	1.667	12	18	10.480	1.720	7	15
		15	250	3.837	0.230	3.288	4.640	15.330	1.985	12	20	11.630	1.963	9	17
		15	500	3.856	0.224	3.440	5.313	15.270	2.201	12	21	12.490	1.789	9	17
		15	1000	3.865	0.145	3.549	4.239	15.550	2.467	12	23	12.580	1.597	9	17

Table 7: Mean and standard deviation of ARL_0 for $\alpha = 0.1$ for each of 100 Phase I samples.

				P		TAX		NTA		IEEE	
Distribution	r	p	N	Mean	S.D.	Mean	S.D.	Mean	S.D.	Mean	S.D.
Lognormal	0	2	100	7.79	2.781	11.677	8.38	10.077	4.167	6.632	2.149
		2	250	9.162	2.012	10.079	2.34	9.909	2.359	8.368	1.836
		2	500	9.383	1.629	9.707	2.099	9.855	1.691	8.986	1.507
		2	1000	9.926	1.317	10.032	1.134	10.073	1.069	9.596	1.562
		15	100	8.264	3.265	8.136	3.192	7.726	2.792	3.022	0.508
		15	250	9.121	1.844	8.993	1.952	8.89	1.886	4.45	0.624
		15	500	9.49	1.895	9.438	1.924	9.394	1.898	5.926	0.678
		15	1000	9.866	1.261	9.803	1.28	9.803	1.32	7.371	0.886
	0.5	2	100	7.816	2.719	10.64	4.831	9.917	3.642	6.69	2.509
		2	250	9.058	1.95	10.024	2.464	9.926	2.413	8.238	1.861
		2	500	9.33	1.699	9.861	2.239	9.733	2.177	9.088	1.219
		2	1000	9.834	1.403	10.076	1.036	9.959	1.318	9.699	1.051
		15	100	8.488	3.247	8.411	2.923	7.905	2.818	3.826	0.762
		15	250	9.488	2.755	9.552	2.755	9.323	2.724	5.458	0.82
		15	500	9.479	2.166	9.509	2.201	9.415	2.076	6.996	1.013
		15	1000	9.924	1.326	9.957	1.434	9.849	1.354	8.35	0.831
Normal	0	2	100	7.397	2.398	39.49	173.5	9.41	4.298	5.784	1.652
		2	250	8.946	1.581	10.077	2.277	9.905	2.094	7.731	1.312
		2	500	9.311	1.747	9.877	2.001	9.704	2.098	8.614	1.284
		2	1000	9.688	1.56	10.043	1.57	10.066	1.456	9.345	1.097
		15	100	7.153	2.53	7.074	2.488	6.863	2.101	2.565	0.296
		15	250	8.433	1.919	8.417	1.898	8.308	1.644	4.177	0.445
		15	500	9.361	1.862	9.355	1.833	9.298	1.837	5.997	0.62
		15	1000	9.625	1.203	9.568	1.194	9.562	1.219	7.483	0.721
	0.5	2	100	7.449	2.343	12.302	27.111	9.237	3.785	6.297	1.656
		2	250	9.187	1.611	9.853	2.553	9.816	2.36	7.922	1.489
		2	500	9.525	1.589	9.89	1.977	9.762	2.205	8.801	1.441
		2	1000	9.882	1.499	9.818	1.596	9.925	1.606	9.567	0.925
		15	100	8.848	3.342	8.576	3.362	8.133	3.257	2.993	0.427
		15	250	9.495	2.365	9.418	2.346	9.176	2.282	4.958	0.629
		15	500	9.769	1.743	9.728	1.763	9.691	1.807	6.755	0.899
		15	1000	10.009	1.103	9.985	1.099	9.87	1.051	8.117	0.833

Table 8: Mean and standard deviation of ARL_0 for $\alpha = 0.01$ for each of 100 Phase I samples.

				P		TAX		NTA		IEEE	
Distribution	r	p	N	Mean	S.D.	Mean	S.D.	Mean	S.D.	Mean	S.D.
Lognormal	0	2	250	19.855	6.813	130.186	375.204	80.564	71.503	13.999	4.455
		2	500	30.546	12.43	108.984	87.752	77.431	50.796	19.786	6.392
		2	1000	50.212	17.419	93.665	47.742	78.622	33.46	32.776	10.193
		15	250	31.794	12.338	30.704	11.684	25.435	8.979	4.491	0.521
		15	500	52.531	25.874	52.154	24.637	45.424	20.342	6.081	0.651
		15	1000	71.838	30.979	70.515	27.112	63.828	24.462	8.122	1.008
	0.5	2	250	19.726	7.767	103.186	146.101	75.116	80.645	13.837	3.823
		2	500	29.761	11.165	93.733	68.13	76.401	47.539	18.101	7.26
		2	1000	47.471	14.639	87.957	41.782	85.724	36.439	30.163	8.943
		15	250	37.136	18.719	39.041	20.919	28.562	12.994	5.534	0.846
		15	500	49.269	24.08	50.814	25.135	43.49	22.339	7.479	1.569
		15	1000	71.234	26.908	72.093	28.962	64.586	24.905	9.943	1.868
Normal	0	2	250	22.179	8.464	200.545	812.723	62.338	53.085	12.087	3.256
		2	500	34.698	14.662	313.366	1162.557	66.259	40.919	19.267	6.156
		2	1000	56.448	17.943	89.088	41.126	83.575	38.183	33.816	8.085
		15	100	12.01	5.198	11.454	4.434	9.453	3.777	2.572	0.301
		15	250	23.308	8.994	23.076	8.472	19.433	7.192	4.203	0.449
		15	500	38.901	19.184	40.418	19.189	35.802	16.31	6.512	0.691
		15	1000	56.505	22.528	57.288	19.406	53.671	16.343	9.859	0.9
	0.5	2	250	23.379	8.277	277.635	927.058	57.263	45.836	13.856	2.847
		2	500	36.253	16.466	200.562	803.045	69.36	43.61	22.139	7.012
		2	1000	58.212	21.491	87.419	44.084	81.107	33.839	38.478	10.591
		15	250	47.729	26.449	51.184	31.633	36.029	21.523	5.148	0.636
		15	500	62.854	36.874	65.686	39.657	51.486	32.833	8.281	1.124
		15	1000	79.513	36.14	82.248	34.151	72.568	34.706	13.1	1.471